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In any ellipse the product of the perpendiculars from the foci on any tangent is constant, *i. e.*, $EF \times E'F' = GF \times G'F' =$ constant.

Hence $E'F'/G'F' = GF/EF =$ constant, and the locus of F' is a fixed line SF'.

Now the lines PS, PF, PN, PF' form an harmonic pencil, and L, F, N, F' are harmonic points.

Hence the lines SL, SF, SN, SF' form an harmonic pencil.

But SL, SF, SF' are fixed. Therefore, SN is a fixed line.

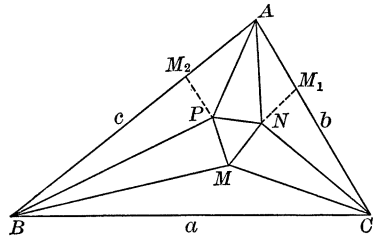
Likewise the foot of the normal corresponding to the other tangent SP' lies on a fixed line which is the harmonic conjugate of SP' with respect to SF and SF'.

431. Proposed by E. M. MORGAN, Dartmouth College.

Trisect the angles of the triangle ABC and let the trisectors nearest each side meet in the respective points M, N, P. Prove by trigonometry that the triangle MNP is equilateral.

SOLUTION BY A. M. HARDING, University of Arkansas.

Let a, b, c denote the sides BC, CA, AB respectively. Construct the points M_1 and M_2 on CA and AB respectively, such that $CM_1 = CM$ and $BM_2 = BM$. Then $MN = M_1N$ and $MP = M_2P$.



If $3\alpha, 3\beta, 3\gamma$ represent the angles A, B, C we have

$$\overline{M_1N}^2 = \overline{AN}^2 + \overline{AM_1}^2 - 2AN \times AM_1 \cos \alpha,$$

or

$$\overline{MN}^2 = \overline{AN}^2 + (b - CM)^2 - 2AN(b - CM) \cos \alpha.$$

Also

$$PN^2 = \overline{AN}^2 + \overline{AP}^2 - 2AN \times AP \cos \alpha.$$

Hence

$$\overline{MN}^2 - \overline{PN}^2 = (b - CM)^2 - \overline{AP}^2 - 2AN(b - CM) \cos \alpha + 2AN \times AP \cos \alpha,$$

$$= (b - CM - AP)(b - CM + AP - 2AN \cos \alpha).$$

Now

$$CM = \frac{\sin \beta}{\sin (\beta + \gamma)} a, \qquad AP = \frac{\sin \beta}{\sin (\alpha + \beta)} c, \qquad AN = \frac{\sin \gamma}{\sin (\alpha + \gamma)} b.$$

Also

$$b = \frac{\sin 3\beta}{\sin 3\alpha} a, \qquad c = \frac{\sin 3\gamma}{\sin 3\alpha} a.$$

Hence

$$b - 2AN\cos\alpha - CM + AP$$

$$= \left[1 - \frac{2\sin\gamma\cos\alpha}{\sin(\alpha+\gamma)}\right]b + \frac{\sin\beta}{\sin(\alpha+\beta)}c - \frac{\sin\beta}{\sin(\beta+\gamma)}a$$

$$= \frac{\sin(\alpha-\gamma)}{\sin(\alpha+\gamma)}b + \frac{\sin\beta}{\sin(\alpha+\beta)}c - \frac{\sin\beta}{\sin(\beta+\gamma)}a$$

$$= \left[\frac{\sin(\alpha-\gamma)\sin 3\beta}{\sin(\alpha+\gamma)\sin 3\alpha} + \frac{\sin\beta\sin 3\gamma}{\sin(\alpha+\beta)\sin 3\alpha} - \frac{\sin\beta}{\sin(\beta+\gamma)}\right]a$$

$$= \left[\frac{\sin(\alpha-\gamma)\sin 3\beta}{\sin(\alpha+\gamma)} + \frac{\sin\beta\sin 3\gamma}{\sin(\alpha+\beta)} - \frac{\sin\beta\sin 3\alpha}{\sin(\beta+\gamma)}\right]\frac{a}{\sin 3\alpha}$$

$$= \{\sin(\alpha-\gamma)[3 - 4\sin^2(\alpha+\gamma)] + \sin\beta[3 - 4\sin^2(\alpha+\beta)]$$
$$- \sin\beta[3 - 4\sin^2(\beta+\gamma)]\}\ a/\sin 3\alpha$$

$$= \{\sin(\alpha-\gamma)[3 - 4\sin^2(\alpha+\gamma)] - 4\sin\beta[\sin^2(\alpha+\beta)$$
$$- \sin^2(\beta+\gamma)]\}\ a/\sin 3\alpha$$

$$= \{\sin(\alpha-\gamma)[3 - 4\sin^2(\alpha+\gamma)] - 2\sin\beta[\cos 2(\beta+\gamma)$$
$$- \cos 2(\alpha+\beta)]\}\ a/\sin 3\alpha$$

$$= \{\sin(\alpha-\gamma)[3 - 4\sin^2(\alpha+\gamma)] - 4\sin\beta[\sin(\alpha-\gamma)\sin(\alpha+2\beta+\gamma)]\}$$
$$a/\sin 3\alpha$$

$$= \sin(\alpha-\gamma)[3 - 4\sin^2(\alpha+\gamma) - 4\sin\beta\sin(\alpha+2\beta+\gamma)]\ a/\sin 3\alpha$$

$$= \sin(\alpha-\gamma)[1 + 2\cos 2(\alpha+\gamma) + 2\cos(\alpha+3\beta+\gamma)$$
$$- 2\cos(\alpha+\beta+\gamma)]\ a/\sin 3\alpha.$$

Now

$$\cos(\alpha+3\beta+\gamma) = -\cos 2(\alpha+\gamma)$$

and

$$\cos(\alpha+\beta+\gamma) = \cos 60^\circ = \tfrac{1}{2}.$$

Hence $b - 2AN\cos\alpha - CM + AP = 0$, and $MN = PN$.

Likewise $MP = PN$.

Also solved by T. M. BLAKESLEE.

CALCULUS.

335. Proposed by W. R. LEBOLD, Cambridge, Ohio.

Let $\rho = F(\theta, \phi)$ be the equation in polar coordinates of a closed surface. Show that the volume of the solid bounded by the surface is equal to the double integral

$$\tfrac{1}{3}\int\int \rho\cos\gamma\, d\sigma$$